International Journal of Theoretical Physics, Vol. 36, No. 7, 1997

Representation of Complex Number Imaginary Unit and Supermatrix Solution to the Yang–Baxter Equation

Yi-Huan Wei[1]

Received October 31, 1996

The double complex number imaginary unit J_D is treated as a special matrix $M(J_D)$; by considering the matrix $M(\epsilon)$ corresponding to the hyperbolic complex imaginary unit ϵ, a supermatrix solution to the Yang–Baxter equation is constructed.

1. INTRODUCTION

Using the ordinary complex imaginary unit i, one can combine a real number pair (x, y) into an ordinary complex number; using the hyperbolic complex imaginary unit ϵ, a real number pair (x, y) can also be combined into a hyperbolic complex number (Yaglom, 1968). Physically, hyperbolic complex numbers were first applied to gravitational fields by Kunstatter *et al.* (1983), and Zhong (1985) gave a hyperbolic complex Ernst equation. In fact, Tanable (1979) used the hyperbolic complex number to construct solutions to the Ernst equation even though the hyperbolic complex imaginary unit ϵ did not appear. By considering the graded representation of the Temper–Lieb algebra, Zhang (1991) constructed the solution to the Yang–Baxter equation (YBE), and Cornwell (1992) derived a two-parameter deformation of the universal enveloping algebra of the Lie algebra $sl(3, c)$ from a multiparameter solution to the YBE; recently, the hyperbolic complex solution to the YBE was constructed from two real solutions by Wu and Zhong (1995).

In this paper, the double complex imaginary unit J_D is treated as a special real double 2×2 matrix $M(J_D)$ and the double complex number $z(J_D)$ corresponds to the real double 2×2 matrix $M[z(J_D)]$; by considering the

[1]Department of Physics, JinZhou Teacher's College, JinZhou 121000, Liaoning, China.

hyperbolic complex imaginary unit ϵ as the 2×2 matrix $M(\epsilon)$, the hyperbolic complex solution to the YBE corresponds to the 2×2 real supermatrix $M(R_H)$ that satisfies the YBE. Furthermore, the 2×2 real supermatrix $M(R_H)$ can be generalized to the 2×2 complex supermatrix.

2. SUBSTITUTION OF $M[(J_D)]$ FOR J_D

Our work starts from the following special 2×2 matrices:

$$M_1 = \begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix}, \qquad M_2 = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \tag{2.1}$$

These matrices M_1 and M_2 appear in many papers (Keller and Rodríguez-Romo, 1991b; MacCallum, 1983; Horowitz and Biedenharn, 1984); we combine the matrices M_1 and M_2 into the double matrix M_D,

$$M_D = \begin{vmatrix} 0 & 1 \\ (-1)^D & 0 \end{vmatrix}, \qquad D = 1, 2 \tag{2.2}$$

where $M_D^2 = M_D M_D = (-1)^D E$ and $E = \mathrm{diag}[1, 1]$ is a 2×2 real unit matrix.

Combining the ordinary complex number imaginary unit i ($i^2 = -1$) and the hyperbolic complex number imaginary unit ϵ ($\epsilon^2 = 1, \epsilon \neq \pm 1$) into the double complex number imaginary unit J_D, one has $J_D^2 = (-1)^D$. By considering the double 2×2 matrix M_D and the double complex number imaginary unit J_D, we have the map of J_D and 1 to the 2×2 matrices M_D and E:

$$M: \quad J_D \to M(J_D) = M_D \tag{2.3}$$

$$M: \quad 1 \to M(1) = E \tag{2.4}$$

Then a complex number $z(J_D) = x + J_D y$ corresponds to a real 2×2 matrix

$$M[z(J_D)] = M(1)x + M(J_D)y \tag{2.5}$$

The conjugate $z^*(J_D) = x - J_D y$ corresponds to

$$M[z^*(J_D)] = M(1)x - M(J_D)y \tag{2.6}$$

Let $|z(J_D)| = (x^2 - J_D^2 y^2)^{1/2}$ be the norm of the complex number $z(J_D)$; then the 'norm' $|M[z(J_D)]|$ of the 2×2 matrix $M[z(J_D)]$ corresponding to the complex number $z(J_D)$ can be defined as

$$|M[z(J_D)]|^2 = M[z(J_D)]M[z^*(J_D)] = |z(J_D)|^2 E \tag{2.7}$$

If there exists the inverse $z^{-1}(J_D)$ of $z(J_D)$, then the corresponding matrix reads

$$M[z^{-1}(J_D)] = |z(J_D)|^{-2} M[z^*(J_D)] = M^{-1}[z(J_D)] \tag{2.8}$$

where $M^{-1}[z(J_D)]$ is the inverse of the matrix $M[z(J_D)]$.

For two double complex numbers $z_1(J_D) = x_1 + J_D y_D$ and $z_2 = x_2 + J_D y_2$, the corresponding matrices $M[z_1(J_D)]$ and $M[z_2(J_D)]$ satisfy

$$M[z_1(J_D)] + M[z_2(J_D)] = M[z_1(J_D) + z_2(J_D)] \tag{2.9}$$

$$M[z_1(J_D)]M[z_2(J_D)] = M[z_1(J_D)z_2(J_D)] \tag{2.10}$$

From the above one can see the set of double complex numbers $z(J_D)$ is isomorphic to the set of double complex matrixes $M[z(J_D)]$.

For the case $D = 2$, we have the following identity (Wu and Zhong 1995):

$$\begin{aligned}&\prod [(\tfrac{1}{2})(x_i + y_i) + (\epsilon/2)(x_i - y_i)]\\ &\quad = \tfrac{1}{2}(\prod x_i + \prod y_i) + (\epsilon/2)(\prod x_i - \prod y_i), \qquad i = 1, 2, \ldots, n\end{aligned} \tag{2.11}$$

We have a similar identity

$$\begin{aligned} M\colon\quad & \{\prod [(1/2)(x_i + y_i) + (\epsilon/2)(x_i - y_i)]\}\\ &= M\{\prod [(1/2)(x_i + y_i) + (\epsilon/2)(x_i - y_i)]\}\\ &= M[(1/2)(\prod x_i + \prod y_i) + (\epsilon/2)(\prod x_i - \prod y_i)]\\ &= (1/2)[E(\prod x_i + \prod y_i) + M(\epsilon)(\prod x_i - \prod y_i)]\end{aligned} \tag{2.12}$$

3. APPLICATION TO YANG–BAXTER EQUATION

According to Wu and Zhong (1995), let A_p and B_q be two real $n \times n$ solutions to the YBE

$$A_{p12}A_{p23}A_{p12} = A_{p23}A_{p12}A_{p23} \tag{3.1a}$$

$$B_{q12}B_{q23}B_{q12} = B_{q23}B_{q12}B_{q23} \tag{3.1b}$$

Then

$$R_{\mathrm{H}} = R_{\mathrm{H}pq} = (1/2)[(A_p + B_q) + \epsilon(A_p - B_q)] \tag{3.2}$$

satisfies the hyperbolic YBE

$$R_{\mathrm{H}12}R_{\mathrm{H}23}R_{\mathrm{H}12} = R_{\mathrm{H}23}R_{\mathrm{H}12}R_{\mathrm{H}23} \tag{3.3}$$

By using (2.3) and (2.4), one can see that the hyperbolic solution R_{H} corresponds to a real 2×2 supermatrix

$$M:\quad R_{\mathrm{H}} \to M(R_{\mathrm{H}}) = (1/2)[E(A_p + B_q) + M(\epsilon)(A_p - B_q)] \tag{3.4}$$

which satisfies the YBE. Direct proof of (3.4) is as follows:

$$\begin{aligned}
&M_{12}(R_{\mathrm{H}})M_{23}(R_{\mathrm{H}})M_{12}(R_{\mathrm{H}})\\
&\quad= \tfrac{1}{8}[E(A_{p12} + B_{q12}) + M(\epsilon)(A_{p12} - B_{q12})][E(A_{p23} + B_{q23})\\
&\qquad + M(\epsilon)(A_{p23} - B_{q23})] \times [E(A_{p12} + B_{q12}) + M(\epsilon)(A_{p12} - B_{q12})]\\
&\quad= \tfrac{1}{4}[E(A_{p12}A_{p23} + B_{q12}B_{q23}) + M(\epsilon)(A_{p12}A_{p23} - B_{q12}B_{q23})]\\
&\qquad \times [E(A_{p12} + B_{q12}) + M(\epsilon)(A_{p12} - B_{q12})]\\
&\quad= \tfrac{1}{2}[E(A_{p12}A_{p23}A_{p12} + B_{q12}B_{q23}B_{q12}) + M(\epsilon)(A_{p12}A_{p23}A_{p12}\\
&\qquad - B_{q12}B_{q23}B_{q12})]\\
&\quad= \tfrac{1}{2}[E(A_{p23}A_{p12}A_{p23} + B_{q23}B_{q12}B_{q23}) + M(\epsilon)(A_{p23}A_{p12}A_{p23}\\
&\qquad - B_{q23}B_{q12}B_{q23})] = M_{23}(R_{\mathrm{H}})M_{12}(R_{\mathrm{H}})M_{23}(R_{\mathrm{H}})
\end{aligned} \tag{3.5}$$

From (3.5) one can find that in the supermatrix (3.4) A_p and B_q can be two complex solutions. For the case in which A_p and B_q are two ordinary complex solutions, $M(R_{\mathrm{H}})$ is an ordinary complex supermatrix solution; for the case in which A_p and B_q are two hyperbolic complex solutions, $M(R_{\mathrm{H}})$ is a hyperbolic complex supermatrix solution.

4. DISCUSSION

(a) We can generalize the double matrix M_D to $M_D(\beta)$,

$$M_D(\beta) = \begin{vmatrix} 0 & \beta^{-1} \\ (-1)^D\beta & 0 \end{vmatrix}, \qquad D = 1, 2 \tag{4.1}$$

where β is an arbitrary real number; then (2.3) and (2.4) read

$$M:\quad J_D \to M(\beta; J_D) = M_D(\beta) \tag{4.2}$$

$$M:\quad 1 \to M(1) = E \tag{4.3}$$

(2.5) and (2.6) read

$$M[\beta; z(J_D)] = M(1)x + M(\beta; J_D)y \tag{4.4}$$

$$M[\beta; z^*(J_D)] = M(1)x - M(\beta; J_D)y \tag{4.5}$$

and (2.7)–(2.10) read

$$|M[\beta; z(J_D)]|^2 = M[\beta; z(J_D)]M[\beta; z^*(J_D)] = |z(J_D)|^2E \tag{4.6}$$

$$M[\beta; z^{-1}(J_D)] = |z(J_D)|^{-2}M[\beta; z^*(J_D)] = M^{-1}[\beta; z(J_D)] \tag{4.7}$$

$$M[\beta; z_1(J_D)] + M[\beta; z_2(J_D)] = M[\beta; z_1(J_D) + z_2(J_D)] \quad (4.8)$$

$$M[\beta; z_1(J_D)]M[\beta; z_2(J_D)] = M[\beta; z_1(J_D)z_2(J_D)] \quad (4.9)$$

For (2.12), we have

$$\begin{aligned} M:\ & \{\prod [(1/2)(x_i + y_i) + (\epsilon/2)(x_i - y_i)]\} \\ &= M(\{\beta; \prod [(1/2)(x_i + y_i) + (\epsilon/2)(x_i - y_i)]\} \\ &= M\{[\beta; (1/2)(\prod x_i + \prod y_i) + (\epsilon/2)(\prod x_i - \prod y_i)] \\ &= (1/2)[E(\prod x_i + \prod y_i) + M(\beta; \epsilon)(\prod x_i - \prod y_i)] \quad (4.10) \end{aligned}$$

(b) We can develop the supermatrix solution (3.4), $M(R_H)$, to $M(\beta; R_H)$:

$$M(\beta; R_H) = (1/2)[E(A_p + B_q) + M(\beta; \epsilon)(A_p - B_q)] \quad (4.11)$$

and we can prove that $M(\beta; R_H)$ satisfies

$$M_{12}(\beta; R_H)M_{23}(\beta; R_H)M_{12}(\beta; R_H) = M_{23}(\beta; R_H)M_{12}(\beta; R_H)M_{23}(\beta; R_H) \quad (4.12)$$

REFERENCES

Cornwell, J. F. (1992). *Zeitschrift für Physik C Particles and Fields*, **56**, 245–252.
Horwitz, L. P., and Biedenharn, L. C. (1984). *Annals of Physics*, **157**, 432–488.
Keller, J., and Rodríguez-Romo, S. (1991). *Journal of Mathematical Physics*, **32**, 1591.
Kunstatter, G., Moffat, J. W., and Malzan, R. (1983). *Journal of Mathematical Physics*, **24**, 866.
MacCallum, M. A. H. (1983). *Journal of Physics A: Mathematical and General*, **16**, 3853–3866.
Tanable, Y. (1979). *Journal of Mathematical Physics*, **20**, 1486.
Wu, Y. B., and Zhong, Z. Z. (1995). *International Journal of Theoretical Physics*, **11**, 2171.
Yaglom, I. M. (1968). *Complex Numbers in Geometry*, Academic Press, London.
Zhang, R. B. (1991). *Journal of Mathematical Physics*, **32**, 2605.
Zhong, Z. Z. (1985). *Journal of Mathematical Physics*, **26**, 2589.